THE INVERSE MEAN CURVATURE FLOW PERPENDICULAR TO THE SPHERE

BEN LAMBERT AND JULIAN SCHEUER

Abstract. We consider the smooth inverse mean curvature flow of strictly convex hypersurfaces with boundary embedded in $\mathbb{R}^{n+1}$, which are perpendicular to the unit sphere from the inside. We prove that the flow hypersurfaces converge to the embedding of a flat disk in the norm of $C^{1,\beta}$, $\beta < 1$.

Contents

1. Introduction

We consider the inverse mean curvature flow in $\mathbb{R}^{n+1}$ with a Neumann boundary condition in a sphere. Let $\mathbb{D} = \mathbb{D}^n$ be the n-dimensional unit disk and $\tilde{N}$ be the outward unit normal of the inclusion $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$. Then we consider a family of embeddings

$$X\colon [0, T^*) \times \mathbb{D} \hookrightarrow \mathbb{R}^{n+1} \tag{1.1}$$

with a normal vector field N, the choice of which will be specified in a natural manner later, such that

$$\dot{X} = \frac{1}{H} N, \tag{1.2a}$$

$$X(\partial\mathbb{D}) = \partial X(\mathbb{D}) \subset \mathbb{S}^n, \tag{1.2b}$$

$$0 = \langle N_{|\partial\mathbb{D}}, \tilde{N}(X_{|\partial\mathbb{D}})\rangle, \tag{1.2c}$$

$$\langle \dot{\gamma}(0), \tilde{N}\rangle \geq 0 \quad \forall \gamma \in C^1((-\epsilon, 0], M_t)\colon \gamma(0) \in \partial X(\mathbb{D}). \tag{1.2d}$$

We prove the following result.

Date: October 20, 2014.
2010 *Mathematics Subject Classification.* 35K61, 53C21, 53C44, 58J32.
Key words and phrases. Curvature flows, inverse curvature flows, Neumann problem, sphere.
The first author is a ZIF-Marie Curie fellow at the Zukunftskolleg, Konstanz.
The second author is being supported by the DFG.

1.1. **Theorem.** *Let*

$$X_0 \colon \mathbb{D} \hookrightarrow M_0 \subset \mathbb{R}^{n+1} \tag{1.3}$$

be the embedding of a smooth and strictly convex hypersurface with normal vector field N_0, such that

$$\langle \dot\gamma(0), \tilde N\rangle \geq 0 \quad \forall \gamma \in C^1((-\epsilon, 0], M_0) \colon \gamma(0) \in \partial X_0(\mathbb{D}), \tag{1.4a}$$

$$X_0(\partial\mathbb{D}) \subset \mathbb{S}^n, \tag{1.4b}$$

$$\langle N_{0|\partial\mathbb{D}}, \tilde N_{|\partial\mathbb{D}}\rangle = 0. \tag{1.4c}$$

Then there exists a finite time $T^* < \infty$, $\alpha > 0$ and a unique solution

$$X \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T^*) \times \mathbb{D}) \cap C^\infty((0, T^*) \times \mathbb{D}, \mathbb{R}^{n+1}) \tag{1.5}$$

of (1.2) with initial hypersurface M_0, such that the embeddings X_t converge to the embedding of a flat unit disk as $t \to T^*$, in the sense that the height of the $M_t = X(t, M)$ over this disk converges to 0.

1.2. *Remark.* The norm of convergence of the M_t to the disk will be specified in Remark 7.4, when we will have developed a suitable coordinate system to describe the M_t.

Our motivation for treating this problem arises from several directions. First of all, the inverse mean curvature flow (IMCF) has proven to be a useful tool in the theory of geometric inequalities, cf. [10] for the probably most famous result in this direction. The works which describe the asymptotic behavior of the IMCF in Euclidean space include [4] and [20], whereas in the hyperbolic space we refer to [7] and [16]. Those works deal with closed hypersurfaces.

Few years ago, the Ph.D. thesis [14] written by Thomas Marquardt appeared, also cf. [15]. Here the IMCF of hypersurfaces with boundary was considered and the embedded flowing hypersurfaces were supposed to be perpendicular to a convex cone in $\mathbb{R}^{n+1}$. However, short-time existence was derived in a much more general situation, in other ambient spaces and with other supporting hypersurfaces besides the cone. It appears to be a natural question, whether one can also obtain nice convergence results if one imposes perpendicularity to other hypersurfaces. Inspired by a recent result about rigidity of hypersurfaces in the sphere by Matthias Makowski and the second author, cf. [13], Oliver Schnürer suggested to the authors that this rigidity result might be helpful to consider the IMCF for hypersurfaces which are perpendicular to the sphere. Indeed, we were able to prove his conjecture that this flow must drive strictly convex hypersurfaces into the embedding of a disk.

The equivalent problem for the mean curvature flow was treated by Axel Stahl in [17] and [18], in which the flow was shown to contract to a point. Other choices of boundary manifolds for a graphical mean curvature flow have shown convergence of the flow to flat disks, see for example [11] and [9], as well as [8] for a levelset approach.

The proof of Theorem 1.1 is ordered as follows: In section 2 we agree on notation and in section 3 we collect the relevant evolution equations and boundary derivatives. In section 4 we make height and gradient estimates for convex hypersurfaces perpendicular to the sphere, which is of interest independently. In particular there follows that if the boundary of a convex manifold is contained in a hemisphere, then we have a lower height bound on the manifold. In section 5 we show that the flow may be written graphically. In section 6 we use the results of section 4 to demonstrate the two key estimates which in conjunction with rigidity results of [13] give the theorem. The first of these is that while the the boundary stays away from an equator, a convex flow has a lower bound on H. The second shows that the flow remains convex up until the singular time. Therefore, due to rigidity at the boundary, ∂M must flow to an equator and so M must flow to a flat disk assuming that the flow may be suitably extended. In section 7 we clarify the necessary PDE existence results and show $C^{1,\beta}$ convergence.

2. Setting and notation

There are various embeddings involved in (1.2), namely the inclusion

$$x\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}, \tag{2.1}$$

the flow embeddings of the form

$$X\colon \mathbb{D} \hookrightarrow \mathbb{R}^{n+1}, \tag{2.2}$$

the inclusion

$$z\colon \partial\mathbb{D} \hookrightarrow \mathbb{D}, \tag{2.3}$$

as well as the derived embedding

$$y\colon \partial\mathbb{D} \hookrightarrow \mathbb{S}^n \tag{2.4}$$

satisfying

$$X \circ z = x \circ y. \tag{2.5}$$

Throughout this paper, we stick to the coordinate based notation for tensors.

Geometric quantities in $\mathbb{R}^{n+1}$ are denoted by a bar, e.g. $(\bar{g}_{\alpha\beta})$ for the Euclidean metric, where greek indices range from 0 to n. We will also write $\langle\cdot,\cdot\rangle$ for the Euclidean scalar product.

Geometric quantities in $\mathbb{S}^n$ are denoted by a check, e.g. $(\check{g}_{ij})$ for the induced metric of the embedding x, where latin indices range from 1 to n.

Induced quantities of embeddings $\mathbb{D} \hookrightarrow \mathbb{R}^{n+1}$ are denoted by latin letters, e.g. the embeddings X induce metrics (g_{ij}), normal vector fields N and a second fundamental forms (h_{ij}), such that we have the Gaussian formula

$$X^\alpha_{ij} = -h_{ij}N^\alpha. \tag{2.6}$$

A hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ is called strictly convex, if N can smoothly be chosen, such that (h_{ij}) is positive definite. For a strictly convex hypersurface we will choose N like this.

Induced quantities of embeddings to $\partial\mathbb{D} \hookrightarrow \mathbb{S}^n$ are denoted by greek letters, e.g. the embeddings y induce metrics (γ_{IJ}), normal vector fields ν and second fundamental forms (η_{IJ}), where capital latin indices range from 2 to n.

Coordinate systems in $\partial\mathbb{D}$ will be denoted by (ξ^I), $2 \le I \le n$.

Define H to be the mean curvature of the embeddings X,

$$H = g^{ij}h_{ij}, \tag{2.7}$$

where (g^{ij}) is the inverse of (g_{ij}).

For an embedded manifold $M^n \hookrightarrow N^{n+1}$ and a function $u\colon M \to \mathbb{R}$, covariant derivatives with respect to the induced metric are denoted by indices, e.g. u_{ij}. If ambiguities are possible, e.g. in the case of tensor derivation, covariant derivatives are denoted by a semicolon, e.g. $h_{ij;k}$. Standard partial derivatives are denoted by a comma, e.g. $u_{i,j}$.

3. Evolution equations and boundary derivatives

For the inverse mean curvature flow the interior evolution equations are well-known. We need the spatial boundary derivatives of various curvature quantities, when the supporting hypersurface is a sphere. The calculations are quite similar to those in [14] and [18]. For the sake of completeness and for a better comprehensibility of the different notation, let us derive them in detail.

3.1. *Remark.* Short-time existence for the flow (1.2) was derived in [14, Thm. 2.12]. Thus we are justified to use (1.2) to calculate the boundary derivatives.

3.2. *Remark.* Due to (1.2c) we obtain that

$$\tilde{N} \in X_*(T\mathbb{D}) \tag{3.1}$$

and thus at boundary points there holds

$$\tilde{n} \equiv (\langle X_k, \tilde{N}\rangle) \in T^{0,1}\mathbb{D}. \tag{3.2}$$

Thus, using (2.5), we see that

$$\mathcal{B} = (\tilde{n}, z_2, \dots, z_n) \tag{3.3}$$

forms a basis of $T_y\mathbb{D}$ for all $y \in \partial\mathbb{D}$. Here we slightly abuse notation and let $\tilde{n}$ denote the contravariant version of $\tilde{n}$ as well. Furthermore we have

$$g_{ij}\tilde{n}^i z_I^j = 0, \quad 2 \le I \le n. \tag{3.4}$$

Boundary derivatives.

3.3. **Lemma.** *On $\partial\mathbb{D}$ there holds*

$$H_i\tilde{n}^i = -H. \tag{3.5}$$

Proof. Note that from

$$\dot{X} = \frac{1}{H}N, \tag{3.6}$$

which also holds on $\partial\mathbb{D}$, we obtain from (2.5) that

$$\frac{1}{H}x_i\nu^i = \frac{1}{H}N = \frac{d}{dt}(X \circ z) = x_i\dot{y}^i, \tag{3.7}$$

where ν denotes the pullback of N along x, which is well defined by (1.2c). We obtain that

$$\dot{y} = \frac{1}{H}\nu \tag{3.8}$$

holds in $T\mathbb{S}^n$. Differentiating (1.2c) with respect to time we obtain

$$\begin{aligned} 0 &= \langle \dot{N}, \tilde{N}\rangle + \langle N, \tilde{N}_i\dot{y}^i\rangle \\ &= \frac{1}{H^2}\langle X_iH^i, \tilde{N}\rangle + \frac{1}{H}\langle N, \check{h}_i^k x_k\nu^i\rangle, \end{aligned} \tag{3.9}$$

which implies the result in view of $\check{h}_i^k = \delta_i^k$. □

3.4. **Lemma.** *On $\partial\mathbb{D}$ there hold*

(i) $h_{ij}\tilde{n}^i z_I^j = 0, \quad 2 \le I \le n,$

(ii) $h_{ij;k}z_I^i z_J^j\tilde{n}^k = -h_{ij}z_I^i z_J^j + h_{ij}\tilde{n}^i\tilde{n}^j g_{kl}z_I^k z_J^l.$

Proof. Differentiating (1.2c) with respect to ξ^I yields, also using (2.5),

$$\begin{aligned} 0 &= \langle N_I, \tilde{N}\rangle + \langle N, \tilde{N}_I\rangle \\ &= h^k_l \tilde{n}_k z^l_I + \langle N, \check{h}^k_l x_k y^l_I\rangle \\ &= h_{ij}\tilde{n}^i z^j_I. \end{aligned} \tag{3.10}$$

Differentiate (2.5) twice and take the scalar product with X_k to obtain

$$z^k_{IJ} = -\check{h}_{lm}\tilde{n}^k y^l_I y^m_J = -\gamma_{IJ}\tilde{n}^k = -g_{ij} z^i_I z^j_J \tilde{n}^k, \tag{3.11}$$

where we used that $\check{h}_{ij} = \breve{g}_{ij}$.

Differentiating (3.10) with respect to ξ^J yields

$$\begin{aligned} h_{ij;k} z^k_J \tilde{n}^i z^j_I &= -h_{ij}\tilde{n}^i_J z^j_I - h_{ij}\tilde{n}^i z^j_{IJ} \\ &= -h_{ij} z^i_J z^j_I + h_{ij}\tilde{n}^i\tilde{n}^j\gamma_{IJ}. \end{aligned} \tag{3.12}$$

□

3.5. **Lemma.** *On $\partial\mathbb{D}$ there holds*

$$h_{ij;k}\tilde{n}^i\tilde{n}^j\tilde{n}^k = -n h_{ij}\tilde{n}^i\tilde{n}^j. \tag{3.13}$$

Proof. With respect to the basis $\mathcal{B}$, g and A split, compare Remark 3.2 and Lemma 3.4. Therefore we have

$$g^{IJ} z^i_I z^j_J = g^{ij} - \tilde{n}^i\tilde{n}^j \tag{3.14}$$

and thus

$$\begin{aligned} -H = H_k\tilde{n}^k &= g^{ij} h_{ij;k}\tilde{n}^k \\ &= h_{ij;k}\tilde{n}^i\tilde{n}^j\tilde{n}^k + h_{ij;k} z^i_I z^j_J \tilde{n}^k g^{IJ} \\ &= h_{ij;k}\tilde{n}^i\tilde{n}^j\tilde{n}^k - h_{ij} z^i_I z^j_J g^{IJ} + h_{ij}\tilde{n}^i\tilde{n}^j g_{kl} z^k_I z^l_J g^{IJ} \\ &= h_{ij;k}\tilde{n}^i\tilde{n}^j\tilde{n}^k - H + n h_{ij}\tilde{n}^i\tilde{n}^j. \end{aligned} \tag{3.15}$$

□

We need another lemma about the induced embedding.

3.6. **Lemma.** *The second fundamental form (η_{IJ}) with respect to the normal $-\nu$ as in (3.8) of the induced embedding*

$$y\colon \partial\mathbb{D} \hookrightarrow \mathbb{S}^n \tag{3.16}$$

satsifies

$$\eta_{IJ} = h_{kl} z^k_I z^l_J. \tag{3.17}$$

In particular, if X is the embedding of a convex hypersurface into $\mathbb{R}^{n+1}$, y is the embedding of a convex hypersurface into the sphere $\mathbb{S}^n$.

Proof. Differentiating (2.5) twice, we obtain from (3.11)

$$\begin{aligned} -x_k\eta_{IJ}\nu^k &= -h_{kl} z^k_I z^l_J N + X_k z^k_{IJ} + \gamma_{IJ}\tilde{N} \\ &= -h_{kl} z^k_I z^l_J N. \end{aligned} \tag{3.18}$$

□

To understand how the height of our hypersurfaces over a hyperplane behaves, we have the following lemma.

3.7. **Lemma.** *Let*

$$X_0\colon \mathbb{D}\to M_0\hookrightarrow \mathbb{R}^{n+1} \tag{3.19}$$

be an embedding as in (1.4). *Let* $\omega\in\mathbb{R}^{n+1}$. *Then the height over the hyperplane* $\omega^\perp$,

$$w=\langle X,\omega\rangle, \tag{3.20}$$

satisfies

$$w_k\tilde{n}^k=w \tag{3.21}$$

on $\partial\mathbb{D}$. *In particular, if* ω *is chosen, such that* w *is positive on* $\partial\mathbb{D}$, w *attains its global minimum in the interior of* $\mathbb{D}$.

Proof. On $\partial\mathbb{D}$ we have

$$\begin{aligned} w_k\tilde{n}^k &= \bar{g}_{\alpha\beta}X^\alpha_k\omega^\beta g^{kl}\bar{g}_{\gamma\delta}X^\gamma_l\tilde{N}^\delta\\ &= \bar{g}_{\beta\delta}\omega^\beta\tilde{N}^\delta\\ &=\langle\tilde{N},\omega\rangle\\ &=w, \end{aligned} \tag{3.22}$$

since on the boundary X maps into $\mathbb{S}^n$ and here the position vector X equals the outer normal $\tilde{N}$. □

Evolution equations. We need the following evolution equations.

3.8. **Lemma.** *The speed*

$$\Phi=-\frac{1}{H} \tag{3.23}$$

satisfies

$$\dot{\Phi}-\frac{1}{H^2}\Delta\Phi=\frac{\|A\|^2}{H^2}\Phi \tag{3.24}$$

in the interior and

$$\Phi_k\tilde{n}^k=\Phi \tag{3.25}$$

on the boundary.

Proof. The interior equation follows from [6, Lemma 2.3.4] and the boundary derivative from Lemma 3.3. □

3.9. **Lemma.** *Let* $\omega\in\mathbb{R}^{n+1}$. *Then the height*

$$w=\langle X,\omega\rangle \tag{3.26}$$

of M_t *over the plane* $\omega^\perp$ *satisfies*

$$\dot{w}-\frac{1}{H^2}\Delta w=\frac{2}{H}\langle N,\omega\rangle \tag{3.27}$$

in the interior and

$$w_k\tilde{n}^k=w \tag{3.28}$$

on the boundary.

Proof. The interior equation comes from (1.2a) and the boundary derivative is derived in Lemma 3.7. □

Applying a strictly convex function in $\mathbb{R}^{n+1}$ to X yields a very useful evolution equation, the derivation of which is a simple calculation.

3.10. **Lemma.** *Let $\chi \in C^2(\mathbb{R}^{n+1})$. Then $\chi = \chi(X)$ satisfies*

$$\dot{\chi} - \frac{1}{H^2}\Delta\chi = \frac{2}{H}\chi_\alpha N^\alpha - \frac{1}{H^2}\chi_{\alpha\beta}X_i^\alpha X_j^\beta g^{ij} \tag{3.29}$$

in the interior and

$$\chi_i \tilde{n}^i = \langle D\chi, \tilde{N}\rangle \tag{3.30}$$

on the boundary.

4. Height estimates

4.1. **Definition.** (i) For a convex hypersurface M_0 satisfying (1.4) let $\operatorname{conv}(\partial M_0)$ denote the convex body in the sphere enclosed by the convex hypersurface $\partial M_0 \hookrightarrow \mathbb{S}^n$, cf. Lemma 3.6.

(ii) For a point $x_0 \in \mathbb{S}^n$, $\mathcal{H}(x_0)$ denotes the closed hemisphere in $\mathbb{S}^n$ with center x_0. The corresponding equator is denoted by $\mathcal{S}(x_0)$.

4.2. **Lemma.** *Let M_0 be a convex hypersurface satisfying (1.4) and*

$$C_0 = \{x \in \mathbb{R}^{n+1} : x = sp, s \geq 0, p \in \operatorname{conv}(\partial M_0)\}. \tag{4.1}$$

Then there holds

$$M_0 \subset C_0. \tag{4.2}$$

Proof. C_0 is a convex cone in $\mathbb{R}^{n+1}$, cf. [3, Prop. 2], and is made of an intersection of half-spaces in $\mathbb{R}^{n+1}$ with normal N_0,

$$C_0 = \bigcap_{y \in \partial M_0} \{x \in \mathbb{R}^{n+1} : \langle x - y, N_0\rangle \leq 0\}. \tag{4.3}$$

The tangent spaces of C_0 and M_0 coincide at all boundary points due to (1.4c) and hence for all boundary points y, M_0 lies on the same side of the tangent plane T_yM_0 as C_0. □

In the sequel we need the following simple geometric lemma.

4.3. **Lemma.** *Let $R > 0$, $e_0 \in \mathbb{R}^{n+1}$ be a unit vector and $C \subset \mathbb{R}^{n+1}$ be a convex closed cone. Then for all $\epsilon > 0$ there exists $\delta > 0$, such that*

$$\langle a, e_0\rangle \geq \cos\left(\frac{\pi}{2} - \epsilon\right)\|a\| \quad \forall a \in C \tag{4.4}$$

implies

$$\langle x, e_0\rangle \geq R + \delta \quad \forall B_R(x) \subset C. \tag{4.5}$$

Proof. Suppose the claim was false. Then there existed $\epsilon > 0$ and a sequence of Euclidean balls $B_R(x_k) \subset C$ with the property

$$R \leq \langle x_k, e_0\rangle < R + \frac{1}{k} \tag{4.6}$$

and such that (4.4) holds. Without loss of generality assume that x_k converges to some $x \in C$. Then we also have

$$B_R(x) \subset C, \tag{4.7}$$

since C is closed. Then

$$a = x - Re_0 \in \bar{B}_R(x) \tag{4.8}$$

and due to (4.4) there holds $a = 0$. Thus we have

$$x = (R, 0, \dots, 0) \tag{4.9}$$

and hence a contradiction to (4.7), since C hits $\{x^0 = 0\}$ at 0 transversally. ☐

4.4. **Lemma.** *Let*

$$X_0 \colon M \hookrightarrow \mathbb{R}^{n+1} \tag{4.10}$$

be the embedding of a strictly convex hypersurface M_0, such that (1.4) holds. Let $e_0 \in \operatorname{int}(\operatorname{conv}(\partial M_0))$ be a direction, such that $\operatorname{conv}(\partial M_0)$ is contained in the open hemisphere $\operatorname{int}(\mathcal{H}(e_0))$. Then we have

$$\varphi := \langle N_0, e_0 \rangle \le C_0 \tag{4.11}$$

for some constant $C_0 < 0$, which only depends on the inradius of $\operatorname{conv}(\partial M_0)$.

Proof. The Gauss map of the embedding X_0,

$$N_0 \colon \mathbb{D} \hookrightarrow \mathbb{S}^n, \tag{4.12}$$

is a diffeomorphism onto its image due to the strict convexity. By Lemma 3.6 and [6, Thm. 9.2.5] the restriction

$$N_{0|\partial\mathbb{D}} \colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n \tag{4.13}$$

is a convex embedding and by [2, Thm. 1.1], there exist two disjoint open connected components A and B, such that

$$\mathbb{S}^n \backslash N_0(\partial\mathbb{D}) = A \cup B \tag{4.14}$$

and A is the interior of the strictly convex body in the sphere, which $N_0(\partial\mathbb{D})$ bounds. Since $\operatorname{conv}(\partial M_0)$ is chosen to be contained in $\mathcal{H}(e_0)$, we have

$$\partial A \subset \mathcal{H}(-e_0) \tag{4.15}$$

and from [6, Thm. 9.2.9, Thm. 9.2.10] we obtain

$$-e_0 \in A \subset \bar{A} \subset \mathcal{H}(-e_0). \tag{4.16}$$

We have either

$$N_0(\mathbb{D}\backslash\partial\mathbb{D}) \subset A \tag{4.17}$$

or

$$N_0(\mathbb{D}\backslash\partial\mathbb{D}) \subset B, \tag{4.18}$$

since the continuous map

$$N_0 \colon \mathbb{D}\backslash\partial\mathbb{D} \to A \cup B \tag{4.19}$$

has to map the connected domain into a connected component, also compare [1, Cor. IV.19.7]. Since the height function

$$w = \langle X, e_0 \rangle \tag{4.20}$$

is increasing at the boundary, cf. Lemma 3.7, it attains an interior minimum and thus $-e_0 \in N_0(\mathbb{D}\backslash\partial\mathbb{D})$. Thus we must have (4.17). This implies the claim. ☐

4.5. **Corollary.** *In the situation of Lemma 4.4 the height function*

$$w = \langle X_0, e_0 \rangle \tag{4.21}$$

does not attain an interior local maximum.

Proof. Using the Gaussian formula we obtain

$$\Delta w = -H\varphi > 0. \tag{4.22}$$

☐

4.6. **Corollary.** *In the situation of Lemma 4.4 there holds*

$$\langle e_0 - X_0, N_0\rangle < 0. \tag{4.23}$$

Proof. Suppose the claim to be false, then there existed a point $z \in \operatorname{int}(\mathbb{D})$ with the property that e_0 is not contained in the supporting open halfspace at $X_0 = X_0(z)$,

$$S_0 = \{x \in \mathbb{R}^{n+1} \colon \langle x - X_0, N_0\rangle < 0\}. \tag{4.24}$$

Due to Lemma 4.4 we then also had

$$0 \notin \bar{S}_0. \tag{4.25}$$

By the strict convexity of M_0 we have

$$X_0(\partial\mathbb{D}) \subset S_0. \tag{4.26}$$

∂S_0 splits $\mathbb{S}^n$ into two spherical caps. Translating the hyperplane ∂S_0 until it hits 0, we see that ∂M_0 originally had to be contained in the spherical cap which is geodesically convex. But by assumption we have $e_0 \in \operatorname{int}(\operatorname{conv}(\partial M_0))$, which contradicts $e_0 \notin S_0$. ☐

We are now able to estimate the height of a hypersurface M_0 as the latter appears in (1.4). It depends on the estimate in Lemma 4.4 and the curvature.

4.7. **Lemma.** *In the situation of Lemma 4.4 the height*

$$w = \langle X, e_0\rangle \tag{4.27}$$

satisfies

$$w \geq \delta > 0, \tag{4.28}$$

for a constant δ, which depends on the constant C_0 in Lemma 4.4, the length of the second fundamental form of M_0 and the distance of ∂M_0 to the equator $\mathcal{S}(e_0)$.

Proof. Let $a \in M_0$ be the interior global minimum point of w. Due to Lemma 4.4 it is possible to write M_0 locally around a as a graph over the unit disk in $\{0\} \times \mathbb{R}^n$, where w is the graph function. Then

$$w_{ij} = -h_{ij}\langle N, e_0\rangle. \tag{4.29}$$

Using [6, Lemma 2.7.6], we obtain that the Hessian of w with respect to Euclidean coordinates only depends on the second fundamental form and on the estimate of $\langle N, e_0\rangle$ from below. Define

$$\hat{M}_0 = \bigcap_{y \in M_0} \{x \in \mathbb{R}^{n+1} \colon \langle x - y, N_0\rangle \leq 0\}. \tag{4.30}$$

From the previous considerations $\hat{M}_0$ satisfies an interior sphere condition at a with interior ball B_R depending on $\sup \|A\|$ and $\langle N, e_0\rangle$. Due to

$$B_R \subset \hat{M}_0 \subset C_0, \tag{4.31}$$

from Lemma 4.3 we obtain the existence of $\delta > 0$, such that

$$\langle a, e_0\rangle \geq \delta. \tag{4.32}$$

☐

4.8. **Corollary.** *In the situation of Lemma 4.4 we have*

$$X_0(\operatorname{int}(\mathbb{D})) \subset \operatorname{int}(B^+), \tag{4.33}$$

where $B^+ \subset \mathbb{R}^{n+1}$ is the pointed halfball

$$B^+ = B_1^+(0) \backslash \{e_0\}. \tag{4.34}$$

Proof. The function

$$\rho = |X_0|^2 \tag{4.35}$$

satisfies

$$\Delta\rho = -2H\langle N_0, X_0\rangle + 2n, \tag{4.36}$$

due to the Gaussian formula. At an interior maximum of ρ we have

$$0 = \nabla\rho \tag{4.37}$$

and thus X_0 has to be a multiple of N_0. Since

$$\langle X_0, e_0\rangle > 0 \tag{4.38}$$

due to Lemma 4.7 and

$$\langle N_0, e_0\rangle < 0 \tag{4.39}$$

due to Lemma 4.4, we have

$$\langle N_0, X_0\rangle < 0. \tag{4.40}$$

Thus at a maximal point we have

$$\Delta\rho > 0, \tag{4.41}$$

a contradiction. Since we have $\rho = 1$ at the boundary, the claim follows. □

5. Moebius coordinates and the scalar flow

In this section we want to derive a scalar flow equation naturally associated with (1.2). Therefore we aim for a graph representation. A natural candidate for hypersurfaces of our type are rotations of Moebius transformations on the plane. Consider a one-parameter familiy of Moebius transformations of the form

$$\tilde{f}(x,\lambda) = \frac{(1+\lambda)x + i(\lambda-1)}{1+\lambda+i(1-\lambda)x}, \tag{5.1}$$

where $(x,\lambda) \in [-1,1]\times[1,\infty)$. For each λ this is a conformal transformation moving the real axis towards i as $\lambda \to \infty$, whereas the boundary of the real interval $[-1,1]$ maps to the unit sphere perpendicularly. A rotation of a plane in $\mathbb{R}^{n+1}$ around the e_0-axis gives rise to the following definition.

5.1. **Definition.** Let $D \subset \mathbb{R}^n$ be the unit disk. Define *Moebius coordinates* for the pointed halfball

$$B^+ := B_1^+(0)\backslash\{e_0\} \tag{5.2}$$

to be the diffeomorphism

$$\begin{aligned} f\colon D \times [1,\infty) &\to B^+ \\ f(x,\lambda) &= \frac{4\lambda x + (1+|x|^2)(\lambda^2-1)e_0}{(1+\lambda)^2 + (1-\lambda)^2|x|^2}. \end{aligned} \tag{5.3}$$

Graphs in Moebius coordinates. Let us provide some general formalae for hypersurfaces $M \subset \mathbb{R}^{n+1}$ which can be written as graphs in Moebius coordinates. Thus suppose the embedding of a hypersurface M is given by a map

$$\begin{aligned} X \colon \mathbb{D} &\hookrightarrow \mathbb{R}^{n+1} \\ z &\mapsto f(x(z), u(x(z))), \end{aligned} \tag{5.4}$$

where $u \colon D \to [1, \infty)$ is a function. First of all, from a tedious computation and the conformality of f we obtain a representation of the Euclidean metric $\delta_{\alpha\beta}$ in Moebius coordinates,

$$d\bar{s}^2 = e^{2\psi}({dx^0}^2 + \sigma_{ij} dx^i dx^j), \tag{5.5}$$

where x^0 corresponds to the λ-coordinate,

$$e^{2\psi} = \left\langle \frac{\partial f}{\partial x^0}, \frac{\partial f}{\partial x^0} \right\rangle, \tag{5.6}$$

$$\frac{\partial f}{\partial \lambda}(x, \lambda) = \frac{(1 + |x|^2)(1 - \lambda^2)}{\lambda((1 + \lambda)^2 + (1 - \lambda)^2 |x|^2)} \left(f - \frac{\lambda^2 + 1}{\lambda^2 - 1} e_0 \right). \tag{5.7}$$

and

$$\sigma_{ij} = e^{-2\psi} \left\langle \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle. \tag{5.8}$$

For M we have the induced metric

$$g_{ij} = e^{2\psi}(u_i u_j + \sigma_{ij}) \tag{5.9}$$

with inverse

$$g^{ij} = e^{-2\psi} \left(\sigma^{ij} - \frac{\sigma^{ik} u_k}{v} \frac{\sigma^{lj} u_l}{v} \right), \tag{5.10}$$

where

$$v^2 = 1 + \sigma^{ij} u_i u_j. \tag{5.11}$$

The contravariant version of the normal is

$$(N^\alpha) = \pm v^{-1} e^{-\psi}(1, -\sigma^{ik} u_k). \tag{5.12}$$

Those formulae can be found in [6, Sec. 1.5].

Due to the conformality of f the outward Euclidean unit normal to D, $\check{N}$, is mapped to a multiple of the unit normal to the sphere in $\mathbb{R}^{n+1}$ which we called $\tilde{N}$ earlier. Thus for a hypersurface satisfying the boundary condition (1.4c) we obtain

$$\begin{aligned} 0 &= \left\langle \check{N}^k \frac{\partial f}{\partial x^k}, N \right\rangle \\ &= \mp \frac{e^\psi}{v} \check{N}^k u_k \end{aligned} \tag{5.13}$$

and thus such a hypersurface satisfies the Neumann boundary condition

$$\check{N}^k u_k = 0. \tag{5.14}$$

Now we prove that hypersurfaces satisfying (1.4) are graphs in Moebius coordinates.

5.2. **Proposition.** *Let*

$$X_0\colon M \hookrightarrow \mathbb{R}^{n+1} \tag{5.15}$$

be the embedding of a strictly convex hypersurface M_0*, such that* (1.4) *holds. Choose* $e_0 \in \operatorname{int}(\operatorname{conv}(\partial M_0))$*, such that* $\operatorname{conv}(\partial M_0)$ *is contained in the open hemisphere* $\mathcal{H}(e_0)$*. Then* M_0 *can be written as a graph in Moebius coordinates around* e_0*, i.e. Moebius coordinates in the pointed half-ball* $B_1^+(0)\backslash\{e_0\}$ *yield a representation*

$$X_0(z) = f(x, u_0(x)), \tag{5.16}$$

where f *is the diffeomorphism defined in* (5.3)*.*

Proof. Due to Corollary 4.8 Moebius coordinates are well-defined throughout M_0. By the implicit function theorem all we have to show is that

$$\left\langle \frac{\partial f}{\partial \lambda}, N_0 \right\rangle < 0. \tag{5.17}$$

Due to Lemma 4.7 we have $\lambda \geq c > 1$ and thus it suffices to discard the negative scalar fraction in (5.7). We have

$$\begin{aligned} \left\langle X_0 - \frac{\lambda^2+1}{\lambda^2-1}e_0, N_0 \right\rangle &= \langle X_0 - e_0, N_0\rangle - \left\langle \frac{2}{\lambda^2-1}e_0, N_0 \right\rangle \\ &> -\frac{2}{\lambda^2-1}\langle e_0, N_0\rangle \\ &> 0, \end{aligned} \tag{5.18}$$

due to Lemma 4.4 and Corollary 4.6. ☐

The previous considerations allow us to naturally associate a scalar parabolic equation to strictly convex solutions of our inverse mean curvature flow (1.2).

5.3. **Corollary.** *Let* X *be a solution of* (1.2) *on a time interval* $[0,\epsilon)$*, such that all* M_t*,* $0 \leq t < \epsilon$*, range within a pointed halfball* B^+ *and are graphs in Moebius coordinates for* B^+*,*

$$M_t = \{(x(t,z), u(t,x))\colon (t,z) \in [0,\epsilon) \times \mathbb{D}\}. \tag{5.19}$$

Then u *solves a parabolic Neumann problem on* $[0,\epsilon) \times D$*, namely*

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{v}{e^{\psi}H} \text{ in } (0,\epsilon) \times D, \\ u_k \breve{N}^k &= 0 \text{ on } [0,\epsilon) \times \partial D, \\ u &= u_0 \text{ on } \{0\} \times D. \end{aligned} \tag{5.20}$$

Proof. For curvature flows in ambient spaces covered by Gaussian coordinate systems the interior equations are deduced in [6, p. 98-99]. Just note that in our case the normal N_0 and the vector $\frac{\partial f}{\partial x^0}$ are pointing in opposite directions, hence the sign. The boundary equation follows from the fact that all M_t are perpendicular to the sphere and by the derivation of (5.14). ☐

6. Curvature estimates and convexity

6.1. *Remark.* Let T^* be the largest time, such that there exists a smooth solution to (1.2) on the interval $[0,T^*)$. This implies mean convexity of M_t, $0 \leq t < T^*$. By Remark 3.1 we indeed have $T^* > 0$. Let $\bar{T} > 0$ be the largest time, such that the solution is smooth on $[0,\bar{T})$ and M_t is strictly convex for all $0 \leq t < \bar{T}$.

6.2. **Proposition.** *Let X be the solution of (1.2) on the interval $[0, \bar{T})$. Then the principal curvatures are bounded, i.e. for $1 \leq i \leq n$ there holds*

$$\kappa_i \leq H \leq \max_{\mathbb{D}} H(0, \cdot) \quad \forall t \in [0, \bar{T}). \tag{6.1}$$

Proof. Using the convexity of the flow hypersurfaces up to $\bar{T}$, all we have to bound is H. From Lemma 3.8 we obtain

$$\dot{H} - \frac{1}{H^2}\Delta H \leq -\frac{\|A\|^2}{H^2} H \tag{6.2}$$

and

$$H_k \tilde{n}^k = -H. \tag{6.3}$$

Thus the claim follows from a standard maximum principle, e.g. [17, Thm. 3.1]. □

6.3. **Lemma.** *On the interval $[0, \bar{T})$ let*

$$y_t \colon \partial\mathbb{D} \hookrightarrow \mathbb{S}^n \tag{6.4}$$

be the induced embeddings of X_t. Then the convex bodies of the embedded submanifolds $\partial M_t \hookrightarrow \mathbb{S}^n$ form an increasing sequence and satisfy uniform interior sphere conditions independently of t.

Proof. The convexity of the ∂M_t in $\mathbb{S}^n$ follow from Lemma 3.6. From (3.8) we see that the enclosed convex bodies are increasing. From Proposition 6.2 and Lemma 3.6 we obtain uniform C^2-estimates and thus uniform interior sphere conditions, also compare [13, Def. 3.2]. □

6.4. **Corollary.** *There exists a $C^{1,\alpha}$ limiting surface $\partial M_{\bar{T}}$ arising as the limit of the ∂M_t. $\partial M_{\bar{T}}$ either is an equator of the sphere or is containd in an open hemisphere.*

Proof. $\partial M_{\bar{T}}$ is the boundary of a weakly convex body in a hemisphere, in the sense of [13, Def. 3.2], also compare [13, Lemma 6.1]. [13, Thm. 1.1] implies the claim. □

We want to conclude that $\bar{T} = T^*$ and that ∂M_{T^*} must be an equator, which would yield the result due to the height estimates. Therefore we need some more estimates.

6.5. **Lemma.** *Let X be the solution of (1.2) on the interval $[0, \bar{T})$ and suppose that $\partial M_{\bar{T}}$ is not an equator. Then there holds*

$$\sup_{[0,\bar{T})\times\mathbb{D}} \frac{1}{H} \leq c, \tag{6.5}$$

where c depends on M_0 and the distance of $\partial M_{\bar{T}}$ to a suitable equator $\mathcal{S}(e_0)$.

Proof. Let $e_0 \in \operatorname{int}(\operatorname{conv}(\partial M_{\bar{T}}))$, such that $\operatorname{conv}(\partial M_{\bar{T}})$ is contained in $\operatorname{int}(\mathcal{H}(e_0))$. Then, due to the monotonicity of $\operatorname{conv}(\partial M_t)$ we also have

$$e_0 \in \operatorname{int}(\operatorname{conv}(\partial M_t)) \tag{6.6}$$

for t close to $\bar{T}$. Thus it is possible to apply Lemma 4.7 to obtain a positive lower bound for the height function

$$w = \langle X_t, e_0 \rangle \geq \delta > 0. \tag{6.7}$$

Define the strictly convex function in $\mathbb{R}^{n+1}$

$$\chi(x) = \frac{1}{2}|\hat{x}|^2 + \frac{\beta}{2}(x^0)^2 - \lambda x^0 + 1, \tag{6.8}$$

where

$$\hat{x} = (0, x^1, \dots, x^n) \tag{6.9}$$

and

$$\lambda > \frac{1}{\delta}, \quad 0 < \beta < 1. \tag{6.10}$$

Define

$$\zeta = \frac{1}{H}\frac{1}{\frac{1}{2} - \chi} \equiv \frac{1}{H}G(\chi). \tag{6.11}$$

Due to the height estimates, ζ is well defined and positive on $[0, \bar{T}) \times \mathbb{D}$. With the help of Lemma 3.8 and Lemma 3.10 a simple computation yields the following evolution equation for ζ, namely

$$\begin{aligned}\dot{\zeta} - \frac{1}{H^2}\Delta\zeta = \frac{\|A\|^2}{H^2}\zeta + 2\chi_\alpha N^\alpha \zeta^2 - \frac{1}{H}\chi_{\alpha\beta}X_i^\alpha X_j^\beta g^{ij}\zeta^2 \\ - 2\chi_i\chi^i\zeta^3 - \frac{2}{H^2}\left(\frac{1}{H}\right)_i G^i\end{aligned} \tag{6.12}$$

and the boundary equation

$$\zeta_i \tilde{n}^i = \left(1 + G\chi_\alpha \tilde{N}^\alpha\right)\zeta. \tag{6.13}$$

Due to $X = \tilde{N}$ on the boundary, we obtain

$$\chi_\alpha \tilde{N}^\alpha = 1 + (\beta - 1)(X^0)^2 - \lambda X^0 \tag{6.14}$$

and thus on the boundary

$$1 + G\chi_\alpha \tilde{N}^\alpha = 1 + \frac{1 + (\beta - 1)(X^0)^2 - \lambda X^0}{\lambda X^0 - \frac{\beta - 1}{2}(X^0)^2 - 1} < 0. \tag{6.15}$$

Now suppose for $0 < T < \bar{T}$ that

$$\max_{[0,T]\times\mathbb{D}} \zeta = \zeta(t_0, z_0) \geq 1, \quad t_0 > 0. \tag{6.16}$$

Then $z_0 \in \operatorname{int}(\mathbb{D})$ and thus from (6.12) we obtain at this point that, also using

$$\frac{G_i}{G} = -\frac{\left(\frac{1}{H}\right)_i}{\frac{1}{H}}, \tag{6.17}$$

$$0 \leq \left(c - \frac{1}{H}\chi_{\alpha\beta}X_i^\alpha X_j^\beta g^{ij}\right)\zeta^2, \tag{6.18}$$

where $c = c(\delta)$. Since

$$\begin{aligned}\chi_{\alpha\beta}X_i^\alpha X_j^\beta g^{ij} &= \chi_{\alpha\beta}\bar{g}^{\alpha\beta} - \chi_{\alpha\beta}N^\alpha N^\beta \\ &= n + \beta - 1 + (1 - \beta)(N^0)^2,\end{aligned} \tag{6.19}$$

we obtain a bound for $\frac{1}{H}$ at the point (t_0, z_0). Since G is bounded, this implies a uniform bound on ζ and in turn a uniform bound on $\frac{1}{H}$. □

6.6. **Proposition.** *There holds $\bar{T} = T^*$. In particular the strict convexity of the flow hypersurfaces is preserved up to T^*.*

Proof. Suppose that $\bar{T} < T^* \leq \infty$. In case that $\partial M_{\bar{T}}$ is an equator of the sphere, we conclude from the height estimates that $M_{\bar{T}}$ is a flat disk and thus a singularity of the flow. This would yield $\bar{T} = T^*$. Thus suppose that $\partial M_{\bar{T}}$ is not an equator. From Lemma 6.5 we obtain

$$\frac{1}{H} \leq c \quad \forall t \in [0, \bar{T}) \tag{6.20}$$

and again the height function satisfies

$$w \geq \delta > 0. \tag{6.21}$$

Define

$$\tilde{H} = \sum_{i=1}^{n} \frac{1}{\kappa_i} = g_{ij}\tilde{h}^{ij}, \tag{6.22}$$

where $(\tilde{h}^{ij})$ is the inverse of (h_{ij}). At a given point choose coordinates with respect to the basis $\mathcal{B} = (\tilde{n}, z_I)$, then at the boundary we deduce, due to Lemma 3.4 and Lemma 3.5, that

$$\begin{aligned} \tilde{H}_k\tilde{n}^k &= -\tilde{h}^r_i\tilde{h}^{si}h_{rs;k}\tilde{n}^k \\ &= -\tilde{h}^1_1\tilde{h}^{11}h_{11;k}\tilde{n}^k - \tilde{h}^J_I\tilde{h}^{KI}h_{JK;k}\tilde{n}^k \\ &= n\tilde{h}^1_1\tilde{h}^{11}h_{11} + \tilde{h}^J_I\tilde{h}^{KI}h_{JK} - \tilde{h}^J_I\tilde{h}^{KI}g_{KJ}h_{11} \\ &\leq (n-1)\tilde{h}^1_1 + \tilde{h}^r_i\tilde{h}^{si}h_{rs} \\ &= (n-1)\tilde{h}^i_j\tilde{n}_i\tilde{n}^j + \tilde{H}. \end{aligned} \tag{6.23}$$

Set

$$\phi = \log \tilde{H} - (n+1)\log w - \alpha t, \quad t < \bar{T}, \tag{6.24}$$

where α will be chosen in dependence of δ and the initial data. From [5, Lemma 6.5] and Lemma 3.9 we obtain

$$\begin{aligned} \dot{\phi} - \frac{1}{H^2}\Delta\phi = &-\frac{\|A\|^2}{H^2} + \frac{2n}{H\tilde{H}} + \frac{2}{H^2\tilde{H}^2}\tilde{H}_i\tilde{H}^i \\ &- \left(\frac{2}{H^2}g^{rs}\tilde{h}^{kl}h_{rk;p}h_{sl;q} - \frac{2}{H^3}H_pH_q\right)\frac{\tilde{h}^{pi}\tilde{h}^q_i}{\tilde{H}} \\ &- \frac{2n+2}{Hw}\langle N, e_0\rangle - \frac{n+1}{H^2w^2}w^iw_i - \alpha \end{aligned} \tag{6.25}$$

in the interior and

$$\phi_k\tilde{n}^k \leq 1 + \frac{n-1}{\tilde{H}}\tilde{h}^1_1 - (n+1) < -1 \quad \forall (t,\xi) \in [0,\bar{T}) \times \partial\mathbb{D}. \tag{6.26}$$

Now suppose that for $0 < T < \bar{T}$ we have

$$\sup_{[0,T]\times\mathbb{D}} \phi = \phi(t_0, z_0), \quad t_0 > 0. \tag{6.27}$$

Then z_0 does not lie on $\partial\mathbb{D}$. From (6.25) we obtain at (t_0, z_0), also using

$$\frac{\tilde{H}_i}{\tilde{H}} = (n+1)\frac{w_i}{w} \tag{6.28}$$

and that the big bracket is nonnegative by [5, equ. (1.7)], that

$$0 \leq c + \frac{2(n+1)^2}{H^2w^2}\|Dw\|^2 - \alpha, \tag{6.29}$$

where the constant depends on δ and the bound on H^{-1}. For large α this is a contradiction. Thus under the assumption that $\partial M_{\bar{T}}$ is not an equator we obtain that the supremum of ϕ would be decreasing and thus ϕ was bounded up to $\bar{T}$. But then

$$\log \tilde{H} = \phi + (n+1)\log w + \alpha t \leq c + \alpha \bar{T}, \tag{6.30}$$

which contradicts the definition of $\bar{T}$, at which $\tilde{H}$ would have to blow up, provided $\bar{T} < T^*$. $\square$

6.7. **Corollary.** *There holds*

$$T^* < \infty. \tag{6.31}$$

Proof. Let $e_0 \in \operatorname{int}(\operatorname{conv}(\partial M_{T^*}))$, such that $\operatorname{conv}(\partial M_{T^*}) \subset \mathcal{H}(e_0)$. The induced strictly convex hypersurfaces $\partial M_t \hookrightarrow \mathbb{S}^n$ satisfy the flow equation (3.8), which has a uniformly positive speed in normal direction. Thus ∂M_{T^*} is reached in finite time. $\square$

7. Convergence to a flat disk

We have seen that as long as the boundary of the flow is strictly contained in an open hemisphere, we have uniform bounds on the height, the mean curvature and the principal curvatures. We want to conclude that the flow can be extended whenever ∂M_{T^*} is not an equator. This would finish the proof of the main result due to the definition of T^*. In this section we will apply regularity theory to the scalar flow equation in Corollary 5.3 to achieve this.

A straightforward computation yields the following representation of this equation.

7.1. **Proposition.** *The function $u\colon (0,T^*) \times D \to [1,\infty)$ satisfies the equation*

$$\frac{\partial u}{\partial t} = -\frac{v}{e^{2\psi}v^{-1}g^{ij}u_{i,j} + A(x,u,Du)} \equiv F(x,u,Du,D^2u), \tag{7.1}$$

where A is smooth and F is a uniformly parabolic operator, provided ∂M_{T^*} is not an equator of the sphere.

Proof. An easy computation gives a relation between covariant and partial derivatives of u, namely

$$u_{ij} = u_{i,j}v^{-2} + r_{ij}(x,u,Du), \tag{7.2}$$

where r_{ij} is a smooth tensor of the indicated variables. Due to [6, equ. (1.5.10)] we obtain

$$h_{ij}v^{-1}\psi^{-1} = u_{i,j}v^{-2} + r_{ij}(x,u,Du) \tag{7.3}$$

with a possibly different, but still smooth, tensor r_{ij}. Inserting this into (5.20) gives the first equality.

The parabolicity follows from

$$\frac{\partial F}{\partial u_{i,j}} = \frac{v}{e^{\psi}H^2}\frac{\partial H}{\partial u_{i,j}} = \frac{1}{H^2}g^{ij}, \tag{7.4}$$

since as long as ∂M_{T^*} is not an equator, we have $H \geq c > 0$ by Lemma 6.5 and g^{ij} is equivalent to the Euclidean metric on D due to (5.18). $\square$

7.2. **Lemma.** *Let* $X\colon (0,T] \to \mathbb{R}^{n+1}$ *be a solution of* (1.2) *and suppose that* ∂M_T *is not an equator of the sphere. Then*

$$T^* > T + \epsilon, \tag{7.5}$$

where ϵ *depends on* M_0 *and the distance of* ∂M_T *to a suitable equator.*

Proof. (i) Considering the scalar proplem as in Corollary 5.3, from Proposition 7.1 and standard regularity theory we obtain C^∞-estimates up to T for u, compare for example [12, Thm. 14.23] or [19, Thm. 4, Thm. 5]. A slight adjustment of the proof of [6, Thm. 2.5.7] to the Neumann boundary case[1] yields a short-time existence interval of length ϵ for $C^{2,\alpha}$ initial functions, depending on the data of the differential operator. In our situation, these data are uniformly under control, such that choosing a flow hypersurface M_{t_0} with $T - t_0 < \epsilon$ yields an extension beyond T. By the standard method of difference quotients this extension is smooth. Thus we have extended the scalar function u.

(ii) To obtain the full curvature flow from the scalar function u, we use the standard method applied in [14, Sec. 2.3], solving an ODE to allow for normal directed evolution. □

Together with Corollary 6.7 and the C^2-estimates we obtain the final result.

7.3. **Corollary.** ∂M_{T^*} *is an equator of the sphere and* M_{T^*} *is an embedded flat disk.*

7.4. *Remark.* From Proposition 6.2 and (7.3) we obtain uniform C^2-bounds for the graph functions u and thus the norm of convergence, in which the flow hypersurfaces converge to unit disk can be characterized by saying that the functions u converge to the constant function with value 1 in the norm of $C^{1,\beta}(D)$.

Acknowledgements. We would like to thank Oliver Schnürer for suggesting the investigation of this interesting problem.

References

[1] Glen Bredon, *Topology and Geometry*, Graduate Texts in Mathematics, Springer, 1993.
[2] Manfredo do Carmo and Frank Warner, *Rigidity and convexity of hypersurfaces in spheres*, J. Differ. Geom. **4** (1970), 133–144.
[3] Orizon Ferreira, Alfredo Iusem, and Sandor Nemeth, *Projections onto convex sets on the sphere*, J. Glob. Optim. **57** (2013), no. 3, 663–676.
[4] Claus Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differ. Geom. **32** (1990), no. 1, 299–314.
[5] ———, *Closed Weingarten hypersurfaces in space forms*, Geom. Anal. Calc. Var. (1996), 71–98.
[6] ———, *Curvature problems*, Series in Geometry and Topology, vol. 39, International Press of Boston Inc., 2006.
[7] ———, *Inverse curvature flows in hyperbolic space*, J. Differ. Geom. **89** (2011), no. 3, 487–527.
[8] Yoshikazu Giga, Masaki Ohnuma, and Moto-Hiko Sato, *On the strong maximum principle and the large time behavior of generalized mean curvature flow with the Neumann boundary condition*, J. Differ. Equ. **154** (1999), no. 1, 107–131.
[9] Gerhard Huisken, *Non-parametric mean curvature evolution with boundary conditions*, J. Differ. Equ. **77** (1989), no. 2, 369–378.
[10] Gerhard Huisken and Tom Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differ. Geom. **59** (2001), no. 3, 353–437.
[11] Ben Lambert, *The constant angle problem for mean curvature flow inside rotational tori*, to appear in Math. Res. Lett. **21**, no. 3, 2014, arxiv:1207.4422.

[1] Also compare www.math.uni-heidelberg.de/studinfo/gerhardt/KleineZeitenNeumann.pdf

[12] Gary Lieberman, *Second order parabolic differential equations*, World Scientific, 1998.

[13] Matthias Makowski and Julian Scheuer, *Rigidity results, inverse curvature flows and Alexandrov-Fenchel type inequalities in the sphere*, preprint, 2013, arxiv:1307.5764v2.

[14] Thomas Marquardt, *The inverse mean curvature flow for hypersurfaces with boundary*, Ph.D. thesis, Freie Universität Berlin, 2012.

[15] ———, *Inverse mean curvature flow for star-shaped hypersurfaces evolving in a cone*, J. Geom. Anal. **23** (2013), no. 3, 1303–1313.

[16] Julian Scheuer, *Non-scale-invariant inverse curvature flows in hyperbolic space*, Calc. Var. Partial Differ. Equ. (2014), online, doi:10.1007/s00526-014-0742-9.

[17] Axel Stahl, *Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differ. Equ. **4** (1996), no. 4, 385–407.

[18] ———, *Convergence of solution to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differ. Equ. **4** (1996), no. 5, 421–441.

[19] Nina Ural'tseva, *A nonlinear problem with an oblique derivative for parabolic equations*, J. Math. Sci. **70** (1994), no. 3, 1817–1827.

[20] John Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), no. 1, 355–372.

Ben Lambert, University of Konstanz, Zukunftskolleg, Box 216, 78457 Konstanz, Germany

E-mail address: `benjamin.lambert@uni-konstanz.de`

Julian Scheuer, Ruprecht-Karls-Universität, Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany

E-mail address: `scheuer@math.uni-heidelberg.de`